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A note on the rate of heat or mass transfer from a small particle freely suspended in a linear shear field

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It is shown that for a small sphere freely suspended in a linear shear flow at small Reynolds numbers, the Nusselt number N is given by $N = \{1 - \alpha P^{\frac{1}{2}} + o(P^{\frac{1}{2}})\}^{-1}$, where P is the Péclet number. For any given type of shear flow, the numerical value of the constant α can be obtained from a general expression derived by Batchelor (1979). The corresponding result for a particle of arbitrary shape is $N/N_0 = \{1 - \alpha N_0 P^{\frac{1}{2}} + O(P^{\frac{1}{2}})\}^{-1}$, where N_0 is the Nusselt number for pure conduction.

1. Introduction

Heat or mass transfer from small solid particles to the surrounding fluid plays an important role in many physical operations, and hence the problem of predicting the rate of such a transfer under a variety of conditions has received a certain amount of attention in the literature.

In the case of small isolated particles, for which the particle Reynolds number is small enough for inertia effects to be negligible, the process is governed by a single dimensionless parameter, the so-called Péclet number P equal to the product of the Reynolds number and the Prandtl number for heat transfer (or the Schmidt number for mass transfer). When P is small, conductive effects predominate over convection and hence the Nusselt number N, i.e. the dimensionless rate of transfer, equals N_0, its value for pure conduction. The additional rate of transfer due to convection can then be obtained as a perturbation in P but, unfortunately, the analysis is not quite as straightforward as might appear at first glance.

As is well known by now, the root of the difficulty lies in the fact that the conductive solution is not uniformly valid throughout the flow field, but applies only within an *inner* region whose dimension relative to the size of the particle is $O(P^{-1})$, if the particle is fixed in a uniform flow, and $O(P^{-\frac{1}{2}})$ if the particle is freely suspended in a linear shear flow. It is necessary, therefore, to develop appropriate inner and outer solutions which must then be matched within the region of overlap.

Using the technique of inner and outer expansions, as first developed by Proudman & Pearson (1957), Acrivos & Taylor (1962) showed that for the case of an isothermal sphere held fixed in an inertialess flow field, uniform at infinity,

$$N = 1 + \tfrac{1}{2}P + \tfrac{1}{2}P^2 \ln P + 0{\cdot}415P^2 + \tfrac{1}{4}P^3 \ln P + O(P^3), \tag{1.1}$$

where

$$N = Q/4\pi ak(T_s - T_\infty) \quad \text{and} \quad P \equiv Ua/\kappa, \tag{1.2}$$

with Q being the rate of heat transfer, a the radius of the sphere, T_s and T_∞, respectively, the temperatures of the sphere and of the ambient fluid, k the thermal conductivity, κ

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the thermal diffusivity, and U the speed of the uniform flow. Brenner (1963) extended (1.1) to a fixed particle of arbitrary shape for which he found that

$$N/N_0 = 1 + \tfrac{1}{2}N_0 P + \tfrac{1}{2}N_0 f P^2 \ln P + O(P^2), \tag{1.3}$$

where $6\pi\mu a U f$, with μ being the viscosity, is the magnitude of the force exerted by the fluid on the particle, and a is half the maximum diameter of the particle.

The reason why such a general result can be derived is that, in the outer region, the velocity appearing in the convective term of the energy equation can be approximated by that of the undisturbed flow with an error which is $O(P)$, because the velocity disturbance due to the particle decays like $O(r^{-1})$ in Stokes flow and is therefore $O(P)$ in the outer region. Consequently, the first term of the outer solution, from which the $O(P)$ correction to N is obtained, is independent of the particle geometry except for a proportionality constant equal to N_0. We wish to remark at this point that, as already stated by Brenner (1963), there exists no unique way of separating the respective contributions of the $P^2 \ln P$ and P^2 terms in (1.3), since the characteristic dimension a in the definition of P is arbitrary.

On the other hand, for the case of a sphere freely suspended in a simple shear flow, Frankel & Acrivos (1968) showed that

$$N = 1 + 0{\cdot}257 P^{\frac{1}{2}} + o(P^{\frac{1}{2}}), \tag{1.4}$$

where $P \equiv \gamma a^2/\kappa$ with γ being the shear rate of the undisturbed flow. Batchelor (1979) extended the analysis to an arbitrary particle freely suspended in a general linear flow and found that

$$N/N_0 = 1 + \alpha N_0 P^{\frac{1}{2}} + o(P^{\frac{1}{2}}), \tag{1.5}$$

where α is a number whose value depends on the definition of P and on the type of linear flow being considered. Batchelor (1979) derived a general expression for determining α – cf. his equation (2.19) – from which he calculated specific values for a two-dimensional and for an axisymmetric pure straining motion in addition to that of a simple shear flow already found by Frankel & Acrivos (1968).

As was the case for a fixed particle in a uniform flow, the additional increase in N due to convection arises to first order – here to $O(P^{\frac{1}{2}})$ – from the solution of the energy equation in which the velocity in the convective term has been set equal to that of the undisturbed linear flow. In contrast to that case, however, the disturbance velocity due to a freely suspended particle decays like r^{-2} and hence, since the appropriate length scale in the outer region is here $O(P^{-\frac{1}{2}})$, the error incurred in the solution of the energy equation in the outer region by neglecting the disturbance velocity is only $O(P^{\frac{3}{2}})$. We should expect, therefore, that one or two additional terms could be added to (1.5) using only the currently available solutions to the energy equation in the two regions. This we shall proceed to do.

We shall deal separately with the cases of a sphere and of a particle of arbitrary shape.

2. Heat transfer from a sphere freely suspended in a steady linear flow

With the origin at the centre of the freely-suspended sphere, we consider an undisturbed velocity field of the form

$$u_i^{(\infty)} = G_{ij} x_j \equiv E_{ij} x_j + \tfrac{1}{2}\epsilon_{ijk}\Omega_j x_k, \quad G_{ii} = E_{ii} = 0, \tag{2.1}$$

where E_{ij} and Ω_j are, respectively, the constant rate of strain tensor and the constant vorticity of the ambient flow. The dimensionless form of the energy equation is

$$\nabla_r^2 T = P u_i \,\partial T/\partial x_i, \tag{2.2}$$

where T is the temperature normalized such that it equals unity at $r = 1$ and vanishes at infinity.

Now, from the work of Frankel & Acrivos (1968) and of Batchelor (1979) we already know that, within the inner region $1 \leqslant r \leqslant O(P^{-\frac{1}{2}})$,

$$T = \frac{1}{r} + \alpha P^{\frac{1}{2}}\left(\frac{1}{r} - 1\right) + O(P),$$

while, within the outer region, $P^{-\frac{1}{2}} \leqslant r \leqslant \infty$,

$$T = P^{\frac{1}{2}} \psi(\rho, x_i/r, E_{ij}, \Omega_k), \quad \rho \equiv P^{\frac{1}{2}} r,$$

where ψ is the fundamental solution of

$$\nabla_\rho^2 \psi = G_{ij} x_j \,\partial\psi/\partial x_i, \tag{2.3}$$

whose asymptotic form as $\rho \to 0$ is given by

$$\psi \to 1/\rho - \alpha + o(\rho). \tag{2.4}$$

However, as remarked earlier, the velocity disturbance due to the presence of the particle is $O(r^{-3})$ smaller than the ambient velocity field, i.e. the error incurred in the outer region on replacing u_i by $G_{ij} x_j$ is $O(P^{\frac{3}{2}})$. Therefore, the temperature in the outer region can be expressed as

$$T = P^{\frac{1}{2}} F(P) \{\psi + O(P^{\frac{3}{2}})\}, \tag{2.5}$$

where $F(P)$ is an as yet unknown function of P, with $F(0) = 1$.

Returning next to the inner region, we see that the appropriate expansion for the temperature T is

$$T = \frac{1}{r} + \alpha P^{\frac{1}{2}}\left(\frac{1}{r} - 1\right) + P T_1 + \alpha P^{\frac{3}{2}} T_2 + o(P^{\frac{3}{2}}),$$

where, in view of (2.2), T_1 and T_2 satisfy

$$\nabla_r^2 T_{1,2} = -\frac{1}{r^3} u_i x_i \tag{2.6}$$

with $T_{1,2} = 0$ at $r = 1$.

Let us now define $\langle T_1 \rangle$ as being the average of T_1 over the surface of a sphere enclosing the particle, i.e.

$$\langle T_1 \rangle \equiv \frac{1}{4\pi} \int T_1 \, d\Omega, \quad r > 1,$$

and similarly for $\langle T_2 \rangle$ and $\langle \psi \rangle$, where $d\Omega$ is here the solid angle. On integrating (2.6) over the surface of a sphere of radius $r > 1$, we therefore obtain, with n_i being the unit outer normal to that sphere, that

$$\frac{1}{r^2}\frac{d}{dr} r^2 \frac{d\langle T_{1,2}\rangle}{dr} = -\frac{1}{4\pi r^2} \int u_i n_i \, d\Omega = 0.$$

Thus, we have within the inner region that

$$\langle T \rangle = 1 + N\left(\frac{1}{r} - 1\right) + o(P^{\frac{3}{2}}), \tag{2.7}$$

where N is the unknown Nusselt number. Moreover, since the above must match with the corresponding average of the outer solution as given by (2.5), we conclude that the $O(\rho)$ and $O(\rho^2)$ terms of (2.4) must vanish when integrated over the surface of a sphere – this can also be proved by an independent argument – so that, within the outer region,

$$\langle T \rangle = P^{\frac{1}{2}}F(P)\{1/\rho - \alpha + o(\rho^2) + O(P^{\frac{3}{2}})\}. \tag{2.8}$$

On matching (2.7) and (2.8) we then obtain that

$$F(P) = N(P) = [1 - \alpha P^{\frac{1}{2}} + o(P^{\frac{3}{2}})]^{-1},$$

i.e.

$$N = 1 + \alpha P^{\frac{1}{2}} + \alpha^2 P + \alpha^3 P^{\frac{3}{2}} + o(P^{\frac{3}{2}}). \tag{2.9}$$

In other words, we have been able to add two more terms to (1.5) for the case of a sphere.

3. Heat transfer from an arbitrary particle freely suspended in a steady linear flow

We shall next consider the case of a non-spherical particle with, however, a sufficient amount of symmetry, for example, an ellipsoid, so that its translational velocity relative to the fluid is still zero. Hence, with the origin of the fixed co-ordinate system at the centre of the particle, the ambient velocity is still given by (2.1). On the other hand, the temperature is now time-dependent, on account of the fact that the particle rotates, so that the energy equation is

$$\nabla_r^2 T = P\{u_i \,\partial T/\partial x_i + \partial T/\partial t\}. \tag{3.1}$$

Let us denote the conductive solution as T_0 which, for $r \gg 1$, has the asymptotic form

$$T_0 = \frac{N_0}{r} + A_{kl}\frac{x_k x_l}{r^5} + O(r^{-4}), \tag{3.2}$$

where A_{kl} is a symmetric, traceless second order tensor which is a function of the time t since the particle orientation continually changes. Of course, N_0 is time-independent.

The inner solution now takes the form

$$T = T_0 + N_0\alpha P^{\frac{1}{2}}(T_0 - 1) + PT_1 + N_0\alpha P^{\frac{3}{2}}T_2 + o(P^{\frac{3}{2}}), \tag{3.3}$$

where $T_{1,2}$ satisfy

$$\nabla^2 T_{1,2} = u_i \,\partial T_0/\partial x_i + \partial T_0/\partial t, \tag{3.4}$$

with $T_{1,2} = 0$ at $r = 1$. In the outer region, the appropriate expression for T is

$$T = N_0 P^{\frac{1}{2}}\{1 + N_0\alpha P^{\frac{1}{2}} + N_0^2\alpha^2 P + o(P)\}\{\psi + P\psi_1 + O(P^{\frac{3}{2}})\}, \tag{3.5}$$

where ψ is as before the fundamental solution of (2.3), while ψ_1 is also a solution of (2.3) with, however, an $O(\rho^{-3})$ singularity at the origin since it must match with the second term of (3.2). Moreover, since the $O(P^{\frac{1}{2}})$ term of the inner solution is harmonic, the $O(\rho^{-2})$ term of ψ_1 must also be harmonic for the two expressions to match, i.e. it must be of the form $B_t x_t/\rho^3$ where B_t is a vector. The latter must, however, be linear in A_{kl} and be a function of E_{ij} and Ω_j, which is clearly impossible. Hence $B_k = 0$, and we conclude that the $O(P^{\frac{1}{2}})$ term of the inner solution is as shown in (3.3).

A particular solution of (3.4) with T_0 given by (3.2) is, for $r \gg 1$,

$$N_0 E_{ij} x_i x_j / 4r + O(r^{-1}) \tag{3.6}$$

and hence the asymptotic form of T_1 which matches with (3.5) and (2.4) is

$$T_1 \to N_0 E_{ij} x_i x_j / 4r - N_0^2 \alpha^2 + O(r^{-1}) \quad \text{for} \quad r \gg 1. \tag{3.7}$$

To determine the $O(P)$ contribution to N, we follow Brenner (1963) and multiply (3.4), for T_1, by T_0 and then integrate the resulting expression over the volume V bounded by a sphere σ of radius $r \gg 1$ and by the particle. On applying the divergence theorem and taking into account the boundary conditions on B, the surface of the particle, and the fact that T_0 is harmonic and equal to unity on B, we obtain that

$$-\frac{1}{4\pi}\int_B n_j \frac{\partial T_1}{\partial x_j} dS = \frac{1}{4\pi}\int_\sigma n_j \frac{\partial T_0}{\partial x_j} T_1 \, dS - \frac{1}{4\pi}\int_\sigma n_j \frac{\partial T_1}{\partial x_j} T_0 \, dS + \frac{1}{8\pi}\int_\sigma u_j n_j T_0^2 \, dS + \frac{1}{8\pi}\frac{\partial}{\partial t}\int_V T_0^2 \, dV. \tag{3.8}$$

On substituting (3.2) and (3.7) we then find for the first integral on the right-hand side of (3.8)

$$\frac{1}{4\pi}\int_\sigma n_j \frac{\partial T_0}{\partial x_j} T_1 \, dS = N_0^3 \alpha^2 + O(r^{-1}),$$

while the second and third integrals are both $O(r^{-1})$. All the integrals that are $O(r^{-k})$, $k \geq 1$, must of course sum up to zero.

The last term in (3.8) can also be written as

$$\frac{1}{8\pi}\frac{d}{dt}\int_V (T_0^2 - N_0^2/r^2)\, dV, \tag{3.9}$$

where V is now the whole volume exterior to the particle. The integral clearly exists and is independent of the particle orientation since it involves only the conduction solution. Thus, the expression in (3.9) will vanish, and, consequently,

$$N/N_0 = 1 + \alpha N_0 P^{\frac{1}{2}} + \alpha^2 N_0^2 P + O(P^{\frac{3}{2}}) = [1 - \alpha N_0 P^{\frac{1}{2}} + O(P^{\frac{3}{2}})]^{-1}. \tag{3.10}$$

It might be tempting at this point to expect that, as was the case for the sphere, the next term in (3.10) would equal $\alpha^2 N_0^3 P^{\frac{3}{2}}$, but, unfortunately, this is not, in general, true. To be sure, although (3.8) applies for T_2 as well, the first two integrals on the right-hand side of (3.8) give rise to two additional $O(1)$ terms which were absent before. Specifically, since T_2 must match with (3.5), we conclude that the $O(\rho^2)$ term in the expansion of ψ must be a second degree harmonic, i.e. it must be of the form $D_{ij} x_i x_j$ where D_{ij} is a symmetric, traceless second order tensor which is a function of E_{ij} and Ω_j. Therefore, the asymptotic expression for T_2 in the overlap region is

$$T_2 \to D_{ij} x_i x_j + N_0 E_{ij} x_i x_j / 4r - N_0^3 \alpha^2 - C + O(r^{-1}),$$

where C is an $O(1)$ constant, linear in A_{kl}, which would arise in general from the expansion of ψ_1 in (3.5) as $\rho \to 0$. The two additional terms are then C and $-\frac{2}{3} D_{jk} A_{kj}$. Both of these are linear in A_{kl} and therefore depend on the instantaneous orientation of the particle, so that to calculate $\bar{N}$, the time-averaged value of N, one would have to obtain the average of A_{kl} by following the motion of the particle. Although, admittedly, this could be achieved for certain specific cases, it does not appear possible at this stage to derive a general result.

We close by briefly considering particles of a more general shape for which the conductive solution T_0 has the form

$$T_0 = \frac{N_0}{r} + \frac{A_k x_k}{r^3} + O(r^{-3}), \tag{3.11}$$

while, owing to the presence of a relative velocity between the particle and the ambient fluid, the undisturbed velocity is now

$$u_i^{(\infty)} = E_{ij} x_j + \tfrac{1}{2}\epsilon_{ijk}\,\Omega_j x_k + U_i, \tag{3.12}$$

where the constant translational velocity U_i is a function of E_{ij} and Ω_j. We wish to ascertain whether the presence of the two vectors A_k and U_i will affect the results we have obtained so far.

First of all, we note that, in the outer region, the appropriate expression for T is, in lieu of (3.5),

$$T = N_0 P^{\frac{1}{2}}\{1 + N_0 \alpha P^{\frac{1}{2}} + \ldots\}\{\psi + P^{\frac{1}{2}}\psi_2 + O(P)\}, \tag{3.13}$$

where ψ_2 satisfies

$$\nabla_\rho^2 \psi_2 = G_{ij} x_j \frac{\partial \psi_2}{\partial x_i} + U_i \frac{\partial \psi}{\partial x_i}, \tag{3.14}$$

i.e. (2.3) with an inhomogeneous term. Clearly ψ_2 is linear in U_i, and because of the matching requirement with T_1, the $O(P)$ term of the inner solution, it must be of the form

$$\psi_2 = \tfrac{1}{2} U_k x_k / r + O(\rho) \quad \text{as} \quad \rho \to 0.$$

That the expression for ψ_2 as $\rho \to 0$ cannot contain an $O(1)$ term independent of x_i can easily be seen by noting that this term, a scalar, must be linear in U_i and be a function only of E_{ij} and Ω_j, an obvious impossibility. Thus, for $r \gg 1$, we obtain in lieu of (3.7)

$$T_1 \to N_0 E_{ij} x_i x_j / 4r + N_0 U_k x_k / 2r - N_0^2 \alpha^2 + O(r^{-1})$$

of which the new term does not contribute to the $O(1)$ integrals on the right-hand side of (3.8). Similarly, all the $O(1)$ integrals involving A_k also vanish, as can easily be verified. We therefore conclude that the expression for N, as given by (3.10), remains valid for all particles.

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